Journal of Applied Mathematics and Mechanics 74 (2010) 198–203

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Wave propagation in a fluid-saturated inhomogeneous porous medium☆

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ARTICLE INFO

Article history:
Received 21 November 2008

ABSTRACT

A closed system of constitutive equations for the dynamical and geometric quantities in a fluid- saturated inhomogeneous elastic porous medium is constructed within the framework of the three-dimensional theory of elasticity. The geometrical characteristics of the wave front and of the ray in a fluid-saturated inhomogeneous medium are obtained from the Fermi's principle.

Steady wave propagation in a homogeneous porous medium has been considered earlier [1–3]. Unsteady elastic waves in a porous homogeneous medium have been investigated [4,5] and the intensity of the wave fronts has been calculated.

Unsteady acceleration waves in a fluid-saturated inhomogeneous porous medium are investigated below. Expressions for of the intensity and geometry of the wave fronts in inhomogeneous porous media are obtained for the first time using the mathematical theory of discontinuities.[6]

1. Velocities of a wave surface

The fluid motion in an elastic porous medium with physicomechanical characteristics which are functions of the coordinates is considered. It is assumed that the pore sizes are small compared with the distance at which the kinematic and geometrical characteristics of the motion change significantly. This enables us to assume that the solid and fluid phases are continuous media and that there will be two displacement vectors at each point of the space: $\mathbf{u}^{(1)}$ is the displacement vector of the solid phase (the skeleton of the porous medium) and $\mathbf{u}^{(2)}$ is the displacement vector of the fluid. The relation between the overall stress tensor and the strain tensor is then written in the form[3–5]

$$\sigma_{ik} = \lambda e_{rr}^{(1)}\delta_{ik} + 2\mu e_{ik}^{(1)} + A_1 e_{rr}^{(2)}\delta_{ik}; \quad P = A_1 e_{kk}^{(1)} + A_2 e_{kk}^{(2)}, \quad A_1 = (1-m)R,$$
$$A_2 = mR \tag{1.1}$$

where $\lambda=\lambda(x_i)$ and $\mu=\mu(x_i)$ are Lamé coefficients, δ_{ik} is the Kronecker delta, P is the force per unit area of the cross section of the porous medium acting on the fluid, $m = m(x_i)$ is the porosity, $R = R(x_i)$ is the compressibility modulus of the fluid, and the superscript 1 refers to the solid phase and the superscript 2 refers to the fluid phase.

Relations (1.1) together with the equations of motion

$$\rho_{11}\ddot{u}_i^{(1)} + \rho_{12}\ddot{u}_i^{(2)} = \sigma_{ik,k}, \quad \rho_{12}\ddot{u}_i^{(1)} + \rho_{22}\ddot{u}_i^{(2)} = P_{,i}; \quad \rho_{11} = \rho_1 - \rho_{12}, \quad \rho_{22} = \rho_2 - \rho_{12} \tag{1.2}$$

and the Cauchy formulae

$$2e_{ij}^{(1)} = u_{i,j}^{(1)} + u_{j,i}^{(1)}, \quad e_{rr}^{(2)} = u_{r,r}^{(2)} \tag{1.3}$$

represent a closed system for the describing the dynamic deformation of an inhomogeneous porous medium. In formulae (1.2) and (1.3), $\rho_{12}=\rho_{12}(x_i)$ is the coefficient of dynamic interaction between the solid phase and the fluid in a pore, $\rho_1=\rho_1(x_i)$ and $\rho_2=\rho_2(x_i)$ are the density of the solid and fluid phases, $\rho_{11} = \rho_{11}(x_i)$ is the effective density of the solid phase, $\rho_{22} = \rho_{22}(x_i)$ is the effective density of the fluid,

☆ *Prikl. Mat. Mekh.* Vol. 74, No. 2, pp. 276-284, 2010.

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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2010.05.010

and $u_k^{(1)}$ and $u_k^{(2)}$ are the components of the displacement of the solid and fluid phases. Summation over repeated Latin indices is carried out from 1 to 3 and, over Greek indices, from 1 to 2. A time derivative is indicated by a dot over the letter.

An acceleration wave in an inhomogeneous porous medium is understood as an isolated surface in which the stresses and rates of displacement of the phases are continuous but certain partial derivatives of them are a discontinuous. The porous medium parameters and their gradients are continuous.

It follows from formulae (1.1) and (1.3) that

$$\dot{\sigma}_{ik} = \lambda \upsilon_{j,j}^{(1)}\delta_{ik} + \mu(\upsilon_{i,k}^{(1)} + \upsilon_{k,i}^{(1)}) + A_1\upsilon_{j,j}^{(2)}\delta_{ik}, \quad \dot{P} = A_1\upsilon_{k,k}^{(1)} + A_2\upsilon_{k,k}^{(2)}, \quad \upsilon_i^{(1)} = \dot{u}_i^{(1)}, \; \upsilon_i^{(2)} = \dot{u}_i^{(2)} \tag{1.4}$$

We will write the difference between expressions (1.2) and (1.4) on the different sides of the wave surface $\Sigma(t)$ (Ref 6) as

$$[\dot{\sigma}_{ik}] = \lambda[\upsilon_{j,j}^{(1)}]\delta_{ik} + \mu([\upsilon_{i,k}^{(1)}] + [\upsilon_{k,i}^{(1)}]) + A_1[\upsilon_{j,j}^{(2)}]\delta_{ik}, \quad [\dot{P}] = A_1[\upsilon_{k,k}^{(1)}] + A_2[\upsilon_{k,k}^{(2)}]$$

$$\rho_{11}[\dot{\upsilon}_i^{(1)}] + \rho_{12}[\dot{\upsilon}_i^{(2)}] = [\sigma_{ik,k}], \quad \rho_{12}[\dot{\upsilon}_i^{(1)}] + \rho_{22}[\dot{\upsilon}_i^{(2)}] = [P_{,i}] \tag{1.5}$$

We apply the geometrical and kinematic conditions for first order compatibility [6] of the phases

$$[\dot{\upsilon}_i^{(\alpha)}] = -\lambda_i^{(\alpha)}G, \quad [\upsilon_{i,k}^{(\alpha)}] = \lambda_i^{(\alpha)}\nu_k, \quad \alpha = 1,2$$

$$[\sigma_{ik,k}] = s_{ik}\nu_k, \quad [P_{,k}] = \eta\nu_k, \quad [\dot{\sigma}_{ik}] = -s_{ik}G, \quad [\dot{P}] = -\eta G \tag{1.6}$$

where S_{ik}, $\lambda_i^{(\alpha)}$ $(\alpha = 1, 2)$ and η are quantities characterizing the jumps in the first derivatives of the stresses, rates of displacement of the phases and the force, G is the velocity of the motion of a wave surface and ν_i are the components of the unit normal to the surface.

Substituting expressions (1.6) into equalities (1.5) and eliminating the quantities s_{ij} and η, we obtain the system of equations

$$(\lambda + \mu)\lambda_j^{(1)}\nu_i\nu_j + \mu\lambda_i^{(1)} + A_1\lambda_j^{(2)}\nu_i\nu_j = \rho_{11}G^2\lambda_i^{(1)} + \rho_{12}G^2\lambda_i^{(2)}$$

$$A_1\lambda_j^{(1)}\nu_i\nu_j + A_2\lambda_j^{(2)}\nu_i\nu_j = \rho_{12}G^2\lambda_i^{(1)} + \rho_{22}G^2\lambda_i^{(2)} \tag{1.7}$$

Assuming that $\lambda_i^{(\alpha)}\nu_i = \omega_\alpha \neq 0\,(\alpha = 1, 2)$ on the surface $\Sigma(t)$, we multiply each equation of system (1.7) by ν_i and sum over the subscript i. As a result, we obtain a homogeneous system of equations in ω_1 and ω_2 $(G = G_l)$

$$(\Lambda_\alpha - \rho_{1\alpha}G_l^2)\omega_1 + (A_\alpha - \rho_{\alpha 2}G_l^2)\omega_2 = 0, \quad \alpha = 1,2; \quad \Lambda_1 = \lambda + 2\mu, \quad \Lambda_2 = A_1 \tag{1.8}$$

In order that system (1.8) should have a non-zero solution, its determinant must be equal to zero. This condition leads to an equation in the velocity G_l of an irrotational wave $\lambda_i^{(\alpha)}\nu_i = \omega_\alpha \neq 0$

$$(\rho_{11}\rho_{22} - \rho_{12}^2)G_l^4 + (2\rho_{12}A_1 - \rho_{11}A_2 - \rho_{22}\Lambda_1)G_l^2 + A_2\Lambda_1 - A_1^2 = 0 \tag{1.9}$$

It follows from Eq. (1.9) that irrotational waves of two types propagate in an inhomogeneous porous medium and the velocities of these waves are found using the formula

$$G_{l_{1,2}}^2 = [2(\rho_{11}\rho_{22} - \rho_{12}^2)]^{-1}(k_1 \pm \sqrt{k_2^2 - 4k_3k_4})$$

$$k_1 = \rho_{11}A_2 - 2\rho_{12}A_1 + \rho_{22}\Lambda_1, \quad k_2 = \rho_{11}A_2 - \rho_{22}\Lambda_1, \quad k_3 = \rho_{22}A_1 - \rho_{12}A_2, \quad k_4 = \rho_{11}A_1 - \rho_{12}\Lambda_1 \tag{1.10}$$

If $\lambda_i^{(\alpha)}\nu_i = 0\,(\alpha = 1, 2)$ on the surface $\Sigma(t)$, subject to the condition that not all $\lambda_i^{(\alpha)}$ are simultaneously equal to zero, then, from system (1.7), we obtain the following formula for determining the velocity $G = G_l$ of an equivoluminal wave

$$G_t^2 = \frac{\mu\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2} \tag{1.11}$$

Hence, two types of irrotational (longitudinal) waves and an equivoluminal (transverse) wave exist in the inhomogeneous porous medium considered and their velocities are equal to the velocities of the waves in a homogeneous porous medium respectively.[4]

2. Intensity of the irrotational waves

We will now determine the change in the intensity of the irrotational waves. To do this, we differentiate Eqs (1.2) with respect to t and Eqs (1.4) with respect to x_k and take the difference between the resulting expressions on different sides of the wave surface. We obtain

$$\lambda[\upsilon_{j,ij}^{(1)}] + \mu([\upsilon_{i,jj}^{(1)}] + [\upsilon_{j,ij}^{(1)}]) + A_1[\upsilon_{j,ij}^{(2)}] + \lambda_{,i}[\upsilon_{j,j}^{(1)}] + \mu_{,j}([\upsilon_{i,j}^{(1)}] + [\upsilon_{j,i}^{(1)}]) + A_{1,i}[\upsilon_{j,j}^{(2)}] =$$
$$= \rho_{11}[\ddot{\upsilon}_i^{(1)}] + \rho_{12}[\ddot{\upsilon}_i^{(2)}]$$

$$A_1[\upsilon_{j,ij}^{(1)}] + A_2[\upsilon_{j,ij}^{(2)}] + A_{1,i}[\upsilon_{j,j}^{(1)}] + A_{2,i}[\upsilon_{j,j}^{(2)}] = \rho_{12}[\ddot{\upsilon}_i^{(1)}] + \rho_{22}[\ddot{\upsilon}_i^{(2)}] \tag{2.1}$$

We now write the geometrical and kinematic first and second order compatibility conditions [6]

$$[\ddot{\upsilon}_i^{(\alpha)}] = L_i^{(\alpha)}G_l^2 - 2G_l\frac{\delta\lambda_i^{(\alpha)}}{\delta t} - \lambda_i^{(\alpha)}\frac{\delta G_l}{\delta t}$$
$$[\upsilon_{j,ij}^{(\alpha)}] = L_j^{(\alpha)}\nu_i\nu_j + g^{\gamma\beta}\lambda_{j,\gamma}^{(\alpha)}(\nu_i x_{j,\beta} + \nu_j x_{i,\beta}) - \lambda_j^{(\alpha)}g^{\gamma\beta}g^{\sigma\tau}b_{\gamma\sigma}x_{i,\beta}x_{j,\tau}$$
$$[\upsilon_{i,jj}^{(\alpha)}] = L_i^{(\alpha)} - 2\Omega_l\lambda_i^{(\alpha)}, \quad [\upsilon_{i,j}^{(\alpha)}] = \lambda_i^{(\alpha)}\nu_j, \quad \alpha = 1,2 \tag{2.2}$$

Here L_i are quantities characterizing the jumps in the second derivatives of the rates of displacement of the phases, $\Omega_l(t)$ is the mean curvature of the wave surface $\Sigma(t)$ of an irrotational wave, $g^{\alpha\beta}$ are the coefficients of the first quadratic form, $b_{\gamma\sigma}$ are the coefficients of the second quadratic form, $x_{i,\beta}$ are the derivatives of the Cartesian coordinates x_i with respect to the curvilinear coordinates u_β of the wave surface and δ denotes δ differentiation with respect to the time t.[6]

We substitute expressions (2.2) into equalities (2.1), multiply by ν_i and, taking account of the fact that $\nu_i\nu_i = 1$, $\nu_i x_{i,\beta} = 0$ and introducing the notation

$$B_\alpha = \rho_{\alpha 2}G_l^2 - A_\alpha, \quad C_\alpha = \rho_{1\alpha}A_2 - \rho_{\alpha 2}A_1; \quad \alpha = 1,2$$

we obtain

$$L_i^{(1)}\nu_i(\rho_{1\alpha}G_l^2 - \Lambda_1) - 2\rho_{1\alpha}G_l\frac{\delta\omega_1}{\delta t} - \rho_{1\alpha}\frac{\delta G_l}{\delta t}\omega_1 + (2\Omega_l\Lambda_\alpha - \Lambda_{\alpha,i}\nu_i)\omega_1 +$$
$$+ L_i^{(2)}\nu_i B_\alpha - 2\rho_{\alpha 2}G_l\frac{\delta\omega_2}{\delta t} - \rho_{\alpha 2}\frac{\delta G_l}{\delta t}\omega_2 + (2\Omega_l A_\alpha - A_{\alpha,i}\nu_i)\omega_2 = 0; \quad \alpha = 1,2 \tag{2.3}$$

We eliminate $L_i^{(2)}$ from system (2.3) in the standard way, taking account of Eq. (1.9). After some reduction, system of equations (2.3) reduces to a single equation with two unknowns ω_1 and ω_2:

$$\left\{-2G_l^3(\rho_{11}\rho_{22} - \rho_{12}^2) + 2G_lC_1\right\}\frac{\delta\omega_1}{\delta t} + 2G_lC_2\frac{\delta\omega_2}{\delta t} +$$
$$+\{(-\rho_{11}\frac{\delta G_l}{\delta t} + 2\Omega_l\Lambda_1 - \Lambda_{1,i}\nu_i)B_2 - (-\rho_{12}\frac{\delta G_l}{\delta t} + 2\Omega_l A_1 - A_1\nu_i)B_1\}\omega_1 +$$
$$+\{(-\rho_{12}\frac{\delta G_l}{\delta t} + 2\Omega_l A_1 - A_{1,i}\nu_i)B_2 - (-\rho_{22}\frac{\delta G_l}{\delta t} + 2\Omega_l A_2 - A_{2,i}\nu_i)B_1\}\omega_2 = 0 \tag{2.4}$$

Using equality (1.8) when $\alpha = 1$, we eliminate ω_2 from Eq. (2.4). Then, after some reduction, we obtain the equation for the change in the intensity of the irrotational waves in an inhomogeneous medium [6]

$$F_1\frac{\delta W_{1l}}{\delta t} + \left(F_2\frac{\delta G_l}{\delta t} + F_3\right)W_{1l} = 0; \quad W_{1l} = \sqrt{\omega_{1l}\omega_{1l}}, \quad \omega_{1l} = \lambda_i^{(1)}\nu_i, \quad \omega_{1l}\omega_{1l} = \lambda_i^{(1)}\lambda_i^{(1)}\nu_i\nu_i = \lambda_i^{(1)}\lambda_i^{(1)} \tag{2.5}$$

Here,

$$\beta_1 = -C_2(\rho_{12}\Lambda_1 - \rho_{11}A_1)$$

$$D_1 = (\rho_{11}\rho_{12}A_2 + \rho_{12}\rho_{22}\Lambda_1 - 2\rho_{11}\rho_{22}A_1)G_l^2 + \rho_{11}A_1A_2 - 2\rho_{12}A_2\Lambda_1 + \rho_{22}A_1\Lambda_1$$

$$D_{2,i}\nu_i = (\rho_{11}\rho_{12}A_{2,i}\nu_i - 2\rho_{11}\rho_{22}A_{1,i}\nu_i + \rho_{12}\rho_{22}\Lambda_{1,i}\nu_i)G_l^4 +$$
$$+ (2\rho_{11}A_2A_{1,i}\nu_i + 2\rho_{22}\Lambda_1A_{1,i}\nu_i - \rho_{12}A_2\Lambda_{1,i}\nu_i - \rho_{22}A_1\Lambda_{1,i}\nu_i - \rho_{12}\Lambda_1A_{2,i}\nu_i - \rho_{11}A_1A_{2,i}\nu_i)G_l^2 +$$
$$+ A_1A_2\Lambda_{1,i}\nu_i - 2A_2\Lambda_1A_{1,i}\nu_i + A_1\Lambda_1A_{2,i}\nu_i$$

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$$F_1 = -2G_l^3(\rho_{11}\rho_{22} - \rho_{12}^2) + 2G_lC_1 + 2G_lC_2\Gamma_l = \frac{2G_lD_1}{B_1}$$

$$F_2 = -\rho_{11}B_2 - \rho_{12}B_1 + C_2\Gamma_l = \frac{D_1}{B_1}$$

$$F_3 = F_3' + F_3'' + F_3'''$$

$$F_3' = 2\Omega_l\{\Lambda_1B_2 + A_1B_1 - G_l^2C_2\Gamma_l\} = -\frac{2\Omega_lG_l^2D_1}{B_1}, \quad F_3'' = 2G_lC_2\frac{\delta\Gamma_l}{\delta t} = \frac{4G_l^2\beta_1}{B_1}\frac{\delta G_l}{\delta t}$$

$$F_3''' = -\Lambda_{1,i}\nu_iB_2 - A_{1,i}\nu_iB_1 - \{A_{1,i}\nu_iB_2 + A_{2,i}\nu_iB_1\}\Gamma_l = \frac{D_{2,i}\nu_i}{B_1}, \quad \Gamma_l = \frac{\rho_{11}G_l^2 - \Lambda_1}{B_1} \tag{2.6}$$

Here,

$$\beta_1 = -C_2(\rho_{12}\Lambda_1 - \rho_{11}A_l)$$

$$D_1 = (\rho_{11}\rho_{12}A_2 + \rho_{12}\rho_{22}\Lambda_1 - 2\rho_{11}\rho_{22}A_l)G_l^2 + \rho_{11}A_lA_2 - 2\rho_{12}A_2\Lambda_1 + \rho_{22}A_l\Lambda_1$$

$$D_{2,i}\nu_i = (\rho_{11}\rho_{12}A_{2,i}\nu_i - 2\rho_{11}\rho_{22}A_{1,i}\nu_i + \rho_{12}\rho_{22}\Lambda_{1,i}\nu_i)G_l^4 +$$
$$+(2\rho_{11}A_2A_{1,i}\nu_i + 2\rho_{22}\Lambda_1A_{1,i}\nu_i - \rho_{12}A_2\Lambda_{1,i}\nu_i - \rho_{22}A_1\Lambda_{1,i}\nu_i - \rho_{12}\Lambda_1A_{2,i}\nu_i - \rho_{11}A_1A_{2,i}\nu_i)G_l^2 +$$
$$+A_1A_2\Lambda_{1,i}\nu_i - 2A_2\Lambda_1A_{1,i}\nu_i + A_1\Lambda_1A_{2,i}\nu_i$$

Taking account of relations (2.6), we write Eq. (2.5) in the form

$$\frac{\delta W_{1l}}{\delta t} = \left(\Omega_l G_l - \frac{\beta_2}{2G_l}\frac{\delta G_l}{\delta t} - \frac{D_{2,i}\nu_i}{2G_lD_1}\right)W_{1l}, \quad \beta_2 = \frac{A_1D_1 + (4\beta_1 - \rho_{12}D_1)G_l^2}{D_1B_1} \tag{2.7}$$

We will denote the distance along the normals to the surface $\Sigma(t)$ by $s \geq 0$. Then, the δ-derivative of the function W_{1l} can be represented in the form [6]

$$\frac{\delta W_{1l}}{\delta t} = G_l\frac{dW_{1l}}{ds}$$

and we can write the equation for the change in the intensity of the irrotational waves (2.7) in a radial system of coordinates [7]

$$\frac{dW_{1l}}{ds} = \left\{\Omega_l - \frac{1}{2}\left(\beta_2\frac{d\ln G_l}{ds} + \frac{1}{\gamma_1}\frac{dD_2}{ds}\right)\right\}W_{1l}; \quad \gamma_1 = G_l^2D_1; \tag{2.8}$$

3. Intensity of the equivoluminal waves

Differentiating the relations $\lambda_i^{(1)}\upsilon_j = 0$, $\lambda_i^{(1)}\upsilon_j = 0$, which are satisfied on the surface of a equivoluminal wave, with respect to α, we obtain

$$\lambda_{j,\alpha}^{(1)}\nu_j = -\lambda_j^{(1)}\nu_{j,\alpha} = \lambda_j^{(1)}g^{\sigma\tau}b_{\sigma\alpha}x_{j,\tau}, \quad \lambda_{j,\alpha}^{(2)}\nu_j = -\lambda_j^{(2)}\nu_{j,\alpha} = \lambda_j^{(2)}g^{\sigma\tau}b_{\sigma\alpha}x_{j,\tau}$$

We then write the conditions for second order compatibility of the phases in the form [6]

$$[\upsilon_{j,ij}^{(1)}] = L_j^{(1)}\nu_i\nu_j + \lambda_{j,\alpha}^{(1)}g^{\alpha\beta}x_{j,\beta}\nu_i, \quad [\nu_{j,ij}^{(2)}] = L_j^{(2)}\nu_i\nu_j + \lambda_{j,\alpha}^{(2)}g^{\alpha\beta}x_{j,\beta}\nu_i \tag{3.1}$$

Substituting expressions (2.2) and (3.1) into equalities (2.1), after some reduction taking account of equality (1.11), we obtain a differential equation in the radial system of coordinates that determines the change in the intensity of the equivoluminal wave during its propagation

$$\frac{dW_{1t}}{ds} = \left\{\Omega_t - \frac{1}{2}\left(\frac{d\ln G_t}{ds} + \frac{d\ln\mu}{ds}\right)\right\}W_{1t}; \quad W_{1t} = \sqrt{\lambda_i^{(1)}\lambda_i^{(1)}}; \tag{3.2}$$

where Ω_t is the mean curvature of the wave surface of the equivoluminal wave.[6]
It follows from the second equality of (1.7) that

$$W_{2t} = \Gamma_t W_{1t}, \quad \Gamma_t = -\rho_{12}/\rho_{22} \tag{3.3}$$

We write Eqs (2.8) and (3.2) in the unified form

$$\frac{dW_{1p}}{ds} = \left(\Omega_p - \frac{1}{2}(\chi_p + g_p)\right)W_{1p}, \quad p = l,t \tag{3.4}$$

$\left[\chi_l = \beta\frac{d\ln G_l}{ds}, g_l = \frac{1}{\gamma}\frac{dD_2}{ds}\right]$ for the irrotational waves and
$\chi_t = \frac{d\ln G_t}{ds}$, for the equivoluminal wave.

4. The Geometry of the wave fronts

As the unknown function, Eqs (3.4) contain the geometrical invariant $\Omega_p(s)$, the mean curvature of the wave front which changes during the propagation of the surface $\Sigma(t)$ and, consequently, they are not closed.

In order to close Eqs (3.4), it is necessary to obtain equations for the mean curvature. The mean curvature $\Omega_p(s)$ is related to the Gaussian curvature K_p and the first ($g_{\alpha\beta}$) and second ($b^{\alpha\beta}$) quadratic forms of the surface $\Sigma(t)$ by the equations [8,9]

$$\frac{d\Omega_p}{ds} = 2\Omega_p^2 - K_p + (2G_p)^{-1}G_{p,\alpha\beta}g^{\alpha\beta}, \quad \frac{dK_p}{ds} = 2\Omega_pK_p + G_p^{-1}(2\Omega_pg^{\alpha\beta} - b^{\alpha\beta})G_{p,\alpha\beta} \tag{4.1}$$

We find the equations of the trajectory of the ray from F

$$\frac{dg^{\alpha\beta}}{ds}=2b^{\alpha\beta},\quad \frac{dg_{\alpha\beta}}{ds}=-2b_{\alpha\beta}$$
$$\frac{db^{\alpha\beta}}{ds}=g^{\alpha\eta}g^{\beta\delta}[(\ln G_p)_{,\eta\delta}+(\ln G_p)_{,\eta}(\ln G_p)_{,\delta}]+3g_{\eta\delta}b^{\alpha\eta}b^{\beta\delta}$$
$$\frac{db_{\alpha\beta}}{ds}=(\ln G_p)_{,\alpha\beta}+(\ln G_p)_{,\alpha}(\ln G_p)_{,\beta}-g^{\eta\delta}b_{\alpha\eta}b_{\beta\delta} \tag{4.2}$$

We find the equations of the trajectory of the ray from Fermi's principle [7]

$$\frac{d\nu_i}{ds}=-g^{\alpha\beta}(\ln G_p)_{,\alpha}x_{i,\beta},\quad \nu_i=\frac{dx_i}{ds},\quad \frac{dx_{i,\alpha}}{ds}=(\ln G_p)_{,\alpha}-g^{\delta\gamma}b_{\delta\alpha}x_{i,\gamma} \tag{4.3}$$

where the vector $x_{i,\alpha}$ is tangential to the surface $\Sigma(t)$.

The system of equations (4.1) - (4.3), in the case of the specified initial data $\Omega_{0p}, K_{0p}, g_0^{\alpha\beta}$, $b_0^{\alpha\beta}$ and the function $G_p(s)$, has a unique solution.

The invariants $G_{p,\alpha\beta}g^{\alpha\beta}$ and $b^{\alpha\beta}G_{p,\alpha\beta}$ in formulae (4.1) take account of the effect of the curvature and torsion of the ray.

We will first consider the case when these invariants are equal to zero. From relations (4.1), we then obtain closed equations in Ω_p and K_p. Eliminating Ω_p and K_p from Eqs (3.4) and (4.1), in this case we obtain the equation for the change in the intensity of the waves in a homogeneous porous medium.

We now determine the intensity of the waves W_{1p} ($p=l, t$) satisfying Eq. (3.4) and the initial conditions

$$W_{1p}^{(0)}(0)=W_{01p}^{(0)},\quad W_{1p}^{(i)}=0,\quad i=1,2,\ldots \tag{4.4}$$

To do this, we first solve the problem of determining of the mean curvature Ω_p and the Gaussian curvature K_p of the wave surface. We apply the method of successive approximations to formulae (4.1), assuming that

$$\Omega_p=\sum_{n=0}^{\infty}\Omega_p^{(n)},\quad K_p=\sum_{n=0}^{\infty}K_p^{(n)} \tag{4.5}$$

where $\Omega_p^{(n)}$, $K_p^{(n)}$ are approximations of order n.

We choose the magnitude of the gradient $G_p(s)$ as the parameter determining the order of the approximations and we assume that the quantity $d(\ln G_p)/ds$ in Eqs. (4.1) is of the first order of magnitude and that the quantities $(\ln G_p)_{,\alpha}$, $(\ln Gp)_{,\alpha\beta}$ are of the second order of magnitude. A homogeneous medium corresponds to the zeroth order. In the first approximation, account is taken of the rate of change in the inhomogeneity along the ray and, in the second approximation, across the ray.

If $G_{p,\alpha\beta}=0$ in Eqs (4.1), which corresponds to a homogeneous porous medium, we then write the solution for Ω_p and K_p in this case in the form

$$\Omega_p=\frac{\Omega_{0p}-K_{0p}s}{\Psi(s)},\quad K_p=\frac{K_{0p}}{\Psi(s)},\quad \Psi(s)=1-2\Omega_{0p}s+K_{0p}s^2 \tag{4.6}$$

where Ω_{0p} and K_{0p} are the mean and Gaussian curvatures of the wave surface $\Sigma(t)$ from which the distance s is measured.

According to formula (4.6), the wave fronts in a homogeneous porous medium are either developable surfaces K_{0p}=0, Ω_{0p}<0), or surfaces of positive curvature $K_{0p}>0$, $\Omega_{0p}<0$.[6]

We will now consider the case of an inhomogeneous porous medium when the invariants $G_{p,\alpha\beta}g^{\alpha\beta}\neq 0$, $b^{\alpha\beta}G_{p,\alpha\beta}\neq 0$.

It follows from the form of Eqs (4.1) and (4.2) that the invariants $G_{p,\alpha\beta}g^{\alpha\beta}$ and $b^{\alpha\beta}G_{p,\alpha\beta}$ depend solely on the change in the function $\ln G_p$ across the ray and determine the magnitude of the curvature and torsion of the ray.

We will now eliminate the Gaussian curvature k_p from Eqs (4.1). The equation for the mean curvature will then have the form

$$\frac{d^2\Omega_p}{ds^2}-6\Omega_p\frac{d\Omega_p}{ds}+4\Omega_p^3+3G_{p,\alpha\beta}g^{\alpha\beta}\Omega_p=Q,\quad Q=G_{p,\alpha\beta}b^{\alpha\beta}+\frac{1}{2}\frac{d}{ds}(G_{p,\alpha\beta}g^{\alpha\beta})$$

Substituting the values of Ω_p from formula (4.5) and using the method of successive approximations (n = 1, 2), we find that the zeroth approximation $\Omega_p^{(0)}$, $K_p^{(0)}$ has the form (4.6), and $\Omega_p^{(1)}=0$, $K_p^{(1)}=0$. The expression for $\Omega_p^{(2)}$ is determined by the shape of the initial surface (Ω_{0p}, K_{0p}) and is very unwieldy. We shall therefore only consider the case when the wave is initially plane: Ω_{0p}=0, K_{0p}=0. Then,

$$\Omega_p^{(0)}=K_p^{(0)}=\Omega_p^{(1)}=K_p^{(1)}=0,\quad \Omega_p=\Omega_p^{(2)}=\frac{1}{2}\int_0^s N(s)ds-\int_0^s K_p^{(2)}(s)ds$$
$$N=G_{p,\alpha\beta}g^{\alpha\beta},\quad K=K^{(2)}=-\int_0^s M(s)ds,\quad M=G_{p,\alpha\beta}b^{\alpha\beta} \tag{4.7}$$

It will follows from relations (4.7) that the change in the geometry of the wave front in an inhomogeneous porous medium is determined by the invariants M and N, characterizing the change in the function $G_p(s)$ in directions perpendicular to the ray.

We will now determine the intensity of the waves W_{1p} ($p=l, t$) that satisfy Eqs (3.4) and initial conditions (4.4).

We substitute the expressions

$$W_{1p} = W_{1p}^{(0)} + W_{1p}^{(1)} + W_{1p}^{(2)} \tag{4.8}$$

and (4.5) into Eq. (3.4) and, solving by the method of successive approximations, we obtain

$$\frac{dW_{1p}^{(0)}}{ds} = \Omega_p^{(0)} W_{1p}^{(0)}$$
$$\frac{dW_{1p}^{(1)}}{ds} = \Omega_p^{(0)} W_{1p}^{(1)} + \left\{\Omega_p^{(1)} - \frac{1}{2}(\chi_p^{(1)} + g_p^{(1)})\right\} W_{1p}^{(0)}$$
$$\frac{dW_{1p}^{(2)}}{ds} = \Omega_p^{(0)} W_{1p}^{(2)} + \left\{\Omega_p^{(1)} - \frac{1}{2}(\chi_p^{(1)} + g_p^{(1)})\right\} W_{1p}^{(1)} + \Omega_p^{(2)} W_{1p}^{(0)} \tag{4.9}$$

where the notation

$$\chi_l^{(1)} = \beta_2 \frac{d\ln G_l}{ds}, \quad g_l^{(1)} = \frac{1}{\gamma}\frac{dD_2}{ds}, \quad \chi_t^{(1)} = \frac{d\ln G_t}{ds}, \quad g_t^{(1)} = \frac{d\ln\mu}{ds}$$

is introduced for the first approximations. The zeroth approximation for $\Omega_p^{(0)}$ is found from relations (4.6).

We write the solution of Eqs (4.9), with the initial conditions (4.4) and $\Omega_p^{(1)}(0) = 0$, in the form

$$W_{1p}^{(0)} = \frac{W_{01p}^{(0)}}{\sqrt{\Psi}}, \quad W_{1p}^{(1)} = -\frac{W_{01p}^{(0)}}{\sqrt{\Psi}} \int_0^s \{(\chi_p^{(1)}(s) + g_p^{(1)}(s)\} ds \tag{4.10}$$

$$W_{1p}^{(2)} = \frac{W_{01p}^{(0)}}{\sqrt{\Psi}} \left\{ \frac{1}{4} \int_0^s \int_0^s \left(\chi_p^{(1)}(s_1) + g_p^{(1)}(s_1)\right)\left(\chi_p^{(1)}(s_2) + g_p^{(1)}(s_2)\right) ds_1 ds_2 + \int_0^s \Omega_p^{(2)}(s_2) ds_2 \right\} \tag{4.11}$$

The zeroth approximation $W_{1p}^{(0)}$ corresponds to a homogeneous porous medium.

The total intensity of the two irrotational waves and the equivoluminal wave in an inhomogeneous porous medium will be

$$W_p = W_{1p} + W_{2p} = (1 + \Gamma_p) W_{1p}, \quad p = l, t \tag{4.12}$$

The values of Γ_l are found from the last formula of (2.6) and Γ_t from the second formula of (3.3).

Hence, in an inhomogeneous porous medium, the system of equations (3.4), (4.1) and (4.2) determines the change in the intensity of the waves, the mean curvature Ω_p and the Gaussian curvature K_p of the wave front and the ray (4.3) along which it occurs.

It can be concluded from this that, since all the parameters of a porous medium are positive, only the mean curvature Ω_p and the Gaussian curvature K_p determine the conditions for the existence of waves.

Consequently, the shapes of the surfaces $\Sigma(t)$ in an inhomogeneous porous medium are the same as in a homogeneous porous medium but the nature of the change in the intensity depends significantly on the inhomogeneity distribution law.

5. Example

Consider a fluid-saturated inhomogeneous porous medium, characterized by elastic moduli $\lambda(x)$ and $\mu(x)$, a fluid compressibility modulus $R(x)$, a coefficient of dynamic interaction of the solid phase and the fluid $\rho_{12}(x)$, effective densities of the solid phase $\rho_{11}(x)$ and of the fluid $\rho_{22}(x)$, and a porosity $m(x)$.

At the instant $t = 0$, an irrotational wave front propogates along the x axis in the xy plane with a velocity which is determined using formula (1.10).

Since $g^{\alpha\beta} = b^{\alpha\beta} = 0$ and $\Omega_{0l} = K_{0l} = 0$ when $x = 0$, we obtain from Eqs. (4.1) that $\Omega_l = K_l = 0$ and, from Eq. (3.4), we find the relation between the intensity of a irrotational wave and the speed and the physicomechanical characteristics of the porous medium

$$W_l(x) = W_l^0(0) \exp\left\{-\frac{1}{2}\int_0^x \left[\beta_2(x)\frac{d\ln G_l(x)}{dx} + \frac{1}{\gamma_1(x)}\frac{dD_2(x)}{dx}\right] dx\right\}$$

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Translated by E. L. S.